Reducible spectral theory with applications to the robustness of matrices in max algebra

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Abstract

Let $a \oplus b = \max(a,b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. By max-algebra we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. The symbol A^k stands for the k^{th} max-algebraic power of a square matrix A. Let us denote by ε any vector whose every component is $-\infty$. The max-algebraic eigenvalue-eigenvector problem is the following: Given $A \in \overline{\mathbb{R}}^{n \times n}$, find $\lambda \in \overline{\mathbb{R}}$, $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$. Certain problems of industrial production lead to the following task: Given $A \in \overline{\mathbb{R}}^{n \times n}$, is there a k such that $A^k \otimes x$ is a max-algebraic eigenvector of A? If the answer is affirmative for every $x \neq \varepsilon$ then A is called robust. We present a description of the sets of all eigenvalues and eigenvectors for a given matrix A and then derive characterisations of robust matrices.

1 Introduction

Let $a \oplus b = \max(a,b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. Obviously, $-\infty$ plays the role of a neutral element for $\oplus$. Throughout the paper we denote $-\infty$ by ε and for convenience we also denote by the same symbol any vector or matrix whose every component is $-\infty$. If $a \in \mathbb{R}$ then the symbol a^{-1} stands for $-a$.

By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} =$

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$\sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. If A is a square matrix then the iterated product $A \otimes A \otimes ... \otimes A$ in which the symbol A appears k-times will be denoted by A^k. We also denote $\Gamma(A) = A \oplus A^2 \oplus ...$.

A square matrix is called *diagonal*, notation $diag(d_1, ..., d_n)$, if its diagonal entries are $d_1, ..., d_n$ and off-diagonal entries are ε. We also denote $I = diag(0, ..., 0)$. Obviously, $A \otimes I = I \otimes A = A$ whenever A and I are of compatible sizes. By definition $A^0 = I$ for any square matrix A.

An ordered pair $D = (N, F)$ is called a *digraph* if N is a non-empty set (of *nodes*) and $F \subseteq N \times N$ (the set of *arcs*). A sequence $\pi = (v_1, ..., v_p)$ of nodes is called a *path* (in D) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in F$ for all $i = 1, ..., p-1$. The node v_1 is called the *starting node* and v_p the *endnote* of π, respectively. If there is a path in D with starting node u and endnote v then we say that v is *reachable* from u, notation $u \to v$. Thus $u \to u$ for any $u \in N$. As usual a digraph D is called *strongly connected* if $u \to v$ and $v \to u$ for any nodes u, v in D. A path $(v_1, ..., v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, ..., p-1, i \neq j$.

In the rest of the paper $N = \{1, ..., n\}$. The digraph *associated* with $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is

$$D_A = (N, \{(i, j); a_{ij} > \varepsilon\}).$$

The matrix A is called *irreducible* if D_A is strongly connected, *reducible* otherwise. Thus, every 1×1 matrix is irreducible.

The max-algebraic *eigenvalue-eigenvector problem* (briefly *eigenproblem*) is the following:

Given $A \in \overline{\mathbb{R}}^{n \times n}$, *find* $\lambda \in \overline{\mathbb{R}}$, $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ *such that* $A \otimes x = \lambda \otimes x$.

This problem has been studied since the 1960's [10]. One of the motivations was the following analysis of the steady-state behaviour of production systems: Suppose that machines $M_1, ..., M_n$ work interactively and in stages. In each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let $x_i(r)$ denote the starting time of the r^{th}stage on machine i $(i = 1, ..., n)$ and let a_{ij} denote the duration of the operation at which machine M_j prepares the component necessary for machine M_i in the $(r+1)st$ stage $(i, j = 1, ..., n)$. Then

$$x_i(r+1) = \max(x_1(r) + a_{i1}, ..., x_n(r) + a_{in}) \ (i = 1, ..., n; r = 0, 1, ...)$$

or, in max-algebraic notation

$$x(r+1) = A \otimes x(r) \ (r = 0, 1, ...)$$

where $A = (a_{ij})$ is called a *production matrix*. We say that the system reaches a *steady state* if it eventually moves forward in regular steps, that is if for some λ and r_0 we have $x(r+1) = \lambda \otimes x(r)$ for all $r \geq r_0$. Obviously, a steady state is reached immediately if $x(0)$ is an eigenvector of A corresponding to an eigenvalue λ. However, if the choice of a start-time vector is restricted we may need to find out for which vectors a steady state will be reached. A particular task is to characterise those production matrices for which a steady state is reached with any start-time vector. In accordance with the terminology in control theory such matrices are called *robust* and it is the primary objective of the present paper to provide a characterisation of such matrices. Note that this task for irreducible matrices has been solved in [7].

Full solution of the eigenproblem in the case of irreducible matrices has been presented in [11] and [16], see also [22]. A general spectral theorem for reducible matrices was presented in [13] and [3]. In Section 2 we provide a proof of this theorem which enables us to analyse the set of eigenvectors and give answers in Section 3 to some specific questions related to the finiteness of the eigenvectors. These results are compared with those for non-negative matrices in conventional linear algebra. We also show how to efficiently find a basis of the eigenspace corresponding to an eigenvalue. These results are used in Section 4 to provide a characterisation of robustness for reducible matrices, thus completing the solution of this question for all $A \in \overline{\mathbb{R}}^{n\times n}$.

Unless stated otherwise, we assume everywhere in this paper that $n \geq 1$ is an integer, $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$and $\lambda \in \overline{\mathbb{R}}$. Let us define

$$V(A,\lambda) = \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes x\},$$

$$\Lambda(A) = \{\lambda \in \overline{\mathbb{R}}; V(A,\lambda) \neq \{\varepsilon\}\},$$

$$\begin{aligned} V(A) &= \bigcup_{\lambda\in\Lambda(A)} V(A,\lambda), \\ V^+(A,\lambda) &= V(A,\lambda) \cap \mathbb{R}^n, \\ V^+(A) &= V(A) \cap \mathbb{R}^n. \end{aligned}$$

Note that if $A = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and $V(A) = \overline{\mathbb{R}}^n$.

A set $S \subseteq \overline{\mathbb{R}}^n$ is called a *(max-algebraic) subspace* if $u, v \in S, \alpha, \beta \in \overline{\mathbb{R}}$ imply $\alpha \otimes u \oplus \beta \otimes v \in S$. It is easily seen that $V(A,\lambda)$ (the set containing ε and all eigenvectors of A corresponding to λ, if any) is a subspace for all $\lambda \in \overline{\mathbb{R}}$.

Let $S \subseteq \overline{\mathbb{R}}^n$ be a subspace. A vector $v \in \overline{\mathbb{R}}^n$ is called an *extremal in S* if $v = u \oplus w$ for $u, v \in S$ implies $v = u$ or $v = w$. We say that $v_1, ..., v_m \in S$ is a *basis* of S if

1. $v_1, ..., v_m$ are extremals in S and
2. for every $v \in S$ we have $v = \sum_i^{\oplus} \alpha_i \otimes v_i$ for some $\alpha_1, ..., \alpha_m \in \overline{\mathbb{R}}$.

If $\pi = (i_1, ..., i_p)$ is a path in D_A then the *weight* of π is $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$. The symbol $\lambda(A)$ stands for the *maximum cycle mean* of A, that is if D_A has at least one cycle then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{1}$$

where the maximisation is taken over all cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{k} \tag{2}$$

denotes the *mean* of the cycle $\sigma = (i_1, ..., i_k, i_1)$. Note that $\lambda(A)$ remains unchanged if the maximisation in (1) is taken over all elementary cycles. If D_A is acyclic we set $\lambda(A) = \varepsilon$. Various algorithms for finding $\lambda(A)$ exist. One of them is Karp's [18] of computational complexity $O(nm)$ where m is the number of finite entries in A (or, equivalently the number of arcs in D_A).

A is called *definite* if $\lambda(A) = 0$. It is easily seen that $V(\alpha \otimes A) = V(A)$ and $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$ for any $\alpha \in \mathbb{R}$. Hence $\lambda(A)^{-1} \otimes A$ is definite whenever $\lambda(A) > \varepsilon$.

The notation $A = (a_1, ..., a_n)$ means that $a_1, ..., a_n$ are the column vectors of A. If A is definite then $\Gamma(A) = A \oplus A^2 \oplus ... \oplus A^n$ [11]. In this case $\Gamma(A) = (g_{ij})$ is the matrix of the lengths of the longest paths in D_A and so, specifically, if $\Gamma(A) = (g_1, ..., g_n)$ then g_i is the vector of the lengths of the longest paths to node i $(i = 1, ..., n)$ [11]. $\Gamma(A)$ is called a *metric matrix*; it can be found using the Floyd-Warshall algorithm using $O(n^3)$ operations [12].

We also denote $E(A) = \{i \in N; \exists \sigma = (i = i_1, ..., i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$. The elements of $E(A)$ are called *eigen-nodes (of A)*, or *critical nodes*. A cycle σ is called *critical* if $\mu(\sigma, A) = \lambda(A)$. *The critical digraph* of A is the digraph $C(A)$ with the set of nodes N; the set of arcs is the union of the sets of arcs of all critical cycles. It is well known that all cycles in a critical digraph are critical [7]. Two nodes i and j in $C(A)$ are called *equivalent* (notation $i \sim j$) if i and j belong to the same critical cycle of A.

Note that if $\lambda(A) = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and the eigenvectors of A are exactly vectors $(x_1, ..., x_n)^T \in \overline{\mathbb{R}}^n$ such that $x_j = \varepsilon$ whenever the j^{th} column of A is not ε (clearly in this case at least one column of A is ε). We will therefore usually assume that $\lambda(A) > \varepsilon$.

Theorem 1.1 *[11] Suppose $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, \lambda(A) > \varepsilon$ and $\Gamma((\lambda(A))^{-1} \otimes A) = (g_{ij}) = (g_1, ..., g_n)$. Then*

- $i \in E(A) \iff g_{ii} = 0$
- *If $i, j \in E(A)$ then $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ if and only if $i \sim j$.*

If $i, j \in E(A)$ and $g_i = \alpha \otimes g_j$ then g_i and g_j are called *equivalent*. Clearly, $\sim$ constitutes a relation of equivalence in N.

Theorem 1.2 *[1] Suppose $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, \lambda(A) > \varepsilon$ and $\Gamma((\lambda(A))^{-1} \otimes A) = (g_1, ..., g_n)$. Then we obtain a basis of $V(A, \lambda(A))$ by taking exactly one g_j for each equivalence class.*

Being motivated by Theorem 1.2 the vectors g_i, $i \in E(A)$, will be called the *fundamental eigenvectors* of A (FEV).

Theorem 1.3 *(Cuninghame-Green [11], Theorem 25.9) Suppose* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, A \neq \varepsilon$*. Then the following hold:*

1. $V^+(A) \subseteq V(A, \lambda(A))$.
2. $V^+(A) \neq \emptyset$ if and only if $\lambda(A) > \varepsilon$ and in D_A there is
$$(\forall j \in N)(\exists i \in E(A)) j \to i.$$
3. If, moreover, $V^+(A) \neq \emptyset$ then
$$V^+(A) = \{ \sum_{i \in E(A)}^{\oplus} \alpha_i \otimes g_i ; \alpha_i \in \mathbb{R}\}$$
where $\Gamma(\lambda(A)^{-1} \otimes A) = (g_1, ..., g_n)$.

Corollary 1.1 *A irreducible* $\Rightarrow V^+(A) \neq \emptyset$.

As we will see later (Theorem 2.1) in fact $V(A) = V^+(A) \cup \{\varepsilon\} = V(A, \lambda(A))$ and thus $\Lambda(A) = \{\lambda(A)\}$ if A is irreducible. The fact that $\lambda(A)$ is the unique eigenvalue of an irreducible matrix A was proved in [10] and then independently in [22]. The description of $V^+(A)$ for irreducible matrices as given in part 3 of Theorem 1.3 was also proved in [16].

Obviously
$$V^+(A) \cup \{\varepsilon\} = \{\Gamma((\lambda(A))^{-1} \otimes A) \otimes z ; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \text{ for all } j \notin E(A)\}.$$

and also, if non-empty,
$$V^+(A) = \{ \sum_{i \in E^*(A)}^{\oplus} \alpha_i \otimes g_i ; \alpha_i \in \mathbb{R}\}$$
where $E^*(A)$ is any maximal set of indices of non-equivalent FEV of A.

If
$$1 \leq i_1 < i_2 < ... < i_k \leq n, K = \{i_1, ..., i_k\} \subseteq N$$
then $A[K]$ denotes the principal submatrix
$$\begin{pmatrix} a_{i_1 i_1} & ... & a_{i_1 i_k} \\ ... & ... & ... \\ a_{i_k i_1} & ... & a_{i_k i_k} \end{pmatrix}$$
of the matrix $A = (a_{ij})$ and $x[K]$ denotes the subvector $(x_{i_1}, ..., x_{i_k})^T$ of the vector $x = (x_1, ..., x_n)^T \in \overline{\mathbb{R}}^n$.

If $D = (N, E)$ is a digraph and $K \subseteq N$ then $D[K]$ denotes the *induced subgraph* of D, that is
$$D[K] = (K, E \cap (K \times K)).$$
Obviously, $D_{A[K]} = D[K]$.

2 Finding All Eigenvalues

The symbol $A \sim B$ means that A can be obtained from B by a simultaneous permutation of rows and columns. Recall that D_A can be obtained from D_B by a renumbering of the nodes if $A \sim B$. Hence if $A \sim B$ then A is irreducible if and only if B is irreducible.

It is obvious that if $A \otimes x = \lambda \otimes x$ and a matrix B arises from A by a simultaneous permutation of the rows and columns then the same permutation applied to the components of x yields a vector y such that $B \otimes y = \lambda \otimes y$. Hence:

Lemma 2.1 *$\Lambda(A) = \Lambda(B)$ if $A \sim B$ and there is a bijection between $V(A)$ and $V(B)$.*

The following lemma is of a special significance for the rest of the paper.

Lemma 2.2 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, \lambda \in \Lambda(A)$ and $x \in V(A, \lambda)$. If $x \notin V^+(A, \lambda)$ then $n > 1$,*

$$A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

$\lambda = \lambda(A^{(22)})$ and hence A is reducible.

Proof. Permute the rows and columns of A simultaneously so that the vector arising from x by the same permutation of its components is $x' = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$, where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p, x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \leq p < n)$ and $A \sim A' = \begin{pmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{pmatrix}$, where $A^{(11)}$ is $p \times p$. The equality $A' \otimes x' = \lambda \otimes x'$ now yields blockwise:

$$\begin{aligned} A^{(12)} \otimes x^{(2)} &= \varepsilon \\ A^{(22)} \otimes x^{(2)} &= \lambda \otimes x^{(2)} \end{aligned}$$

Since $x^{(2)}$ is finite, it follows from Theorem 1.3 that $\lambda = \lambda(A^{(22)})$; also clearly $A^{(12)} = \varepsilon$. ∎

Theorem 2.1 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then $V(A) = V^+(A)$ if and only if A is irreducible.*

Proof. It remains to prove the "only if" part since the "if" part follows from Lemma 2.2 immediately. If A is reducible then $n > 1$ and $A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$, where $A^{(22)}$ is irreducible. By setting $\lambda = \lambda(A^{(22)}), x^{(2)} \in V^+(A_{22}), x = \begin{pmatrix} \varepsilon \\ x^{(2)} \end{pmatrix} \in \overline{\mathbb{R}}^n$ we see that $x \in V(A) - V^+(A)$. ∎

Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ can be transformed in linear time by simultaneous permutations of the rows and columns to a *Frobenius normal form* (FNF) [20]

$$\begin{pmatrix} A_{11} & \varepsilon & ... & \varepsilon \\ A_{21} & A_{22} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{r1} & A_{r2} & ... & A_{rr} \end{pmatrix} \tag{3}$$

where $A_{11}, ..., A_{rr}$ are irreducible square submatrices of A. If A is in an FNF then the corresponding partition of the node set N of D_A will be denoted as $N_1, ..., N_r$ and these sets will be called *classes (of* A). It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, ..., r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \geq j$. As a slight abuse of language we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of N_j.

If A is in an FNF, say (3), then the *condensation digraph*, notation C_A, is the digraph $(\{N_1, ..., N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\})$.

The symbol $N_i \to N_j$ means that there is a directed path from a node in N_i to a node in N_j in D_A (and therefore from each node in N_i to each node in N_j). Equivalently, there is a directed path from N_i to N_j in C_A.

If there are neither outgoing nor incoming arcs from or to an induced subgraph $C_A[\{N_{i_1}, ..., N_{i_s}\}]$ $(1 \leq i_1 < ... < i_s \leq r)$ and no proper subdigraph has this property then the submatrix

$$\begin{pmatrix} A_{i_1 i_1} & \varepsilon & ... & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{i_s i_1} & A_{i_s i_2} & ... & A_{i_s i_s} \end{pmatrix}$$

is called an *isolated superblock* (or just *superblock*). The induced subdigraph of C_A corresponding to an isolated superblock is a directed tree (though the underlying undirected graph is not necessarily acyclic). C_A is the union of a number of such directed trees. The nodes of C_A with no incoming arcs are called the *initial classes*, those with no outgoing arcs are called the *final classes*. Note that the directed tree corresponding to an isolated superblock may have several initial and final classes.

For instance the condensation digraph for the matrix

$$\begin{pmatrix} A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\ * & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66} \end{pmatrix} \tag{4}$$

can be seen in Figure 1 (note that here and elsewhere $*$ indicates submatrix different from ε). It consists of two superblocks and six classes including three initial and two final ones.

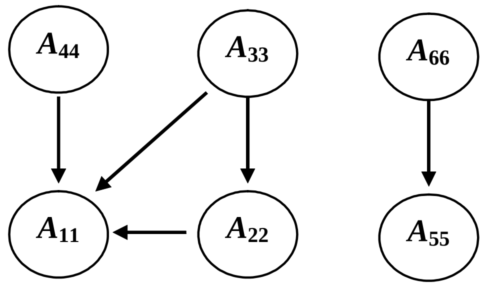

Figure 1: Condensation digraph for matrix (4)

Lemma 2.3 *If $x \in V(A), N_i \to N_j$ and $x[N_j] \neq \varepsilon$ then $x[N_i]$ is finite. In particular, $x[N_j]$ is finite.*

Proof. Suppose that $x \in V(A, \lambda)$ for some $\lambda \in \overline{\mathbb{R}}$. Fix $s \in N_j$ such that $x_s > \varepsilon$. Since $N_i \to N_j$ we have that for every $r \in N_i$ there is a positive integer q such that $b_{rs} > \varepsilon$ where $B = A^q = (b_{ij})$. Since $x \in V(B, \lambda^q)$ we also have $\lambda^q \otimes x_r \geq b_{rs} \otimes x_s > \varepsilon$. Hence $x_r > \varepsilon$. ■

Theorem 2.2 *(Spectral Theorem) Let (3) be an FNF of a matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then*

$$\Lambda(A) = \{\lambda(A_{jj}); \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}.$$

Proof. Note first that

$$\lambda(A) = \max_{i=1,\dots,r} \lambda(A_{ii}) \tag{5}$$

for a matrix A in FNF (3).

Suppose $\lambda(A_{jj}) = \max\{\lambda(A_{ii}); N_i \to N_j\}$ for some $j \in R = \{1, ..., r\}$. Denote $S_2 = \{i \in R; N_i \to N_j\}$, $S_1 = R - S_2$, $M_p = \bigcup_{i \in S_p} N_i$ $(p = 1, 2)$. Then $\lambda(A_{jj}) = \lambda(A[M_2])$ and $A \sim \begin{pmatrix} A[M_1] & \varepsilon \\ * & A[M_2] \end{pmatrix}$.

If $\lambda(A_{jj}) = \varepsilon$ then at least one column, say ℓ^{th} in $A[M_2]$ is ε. We set x_ℓ to any real number and $x_j = \varepsilon$ for $j \neq l$. Then $x \in V(A, \lambda(A_{jj}))$.

If $\lambda(A_{jj}) > \varepsilon$ then $A[M_2]$ has a finite eigenvector by Theorem 1.3, say $\bar{x}$. Set $x[M_2] = \bar{x}$ and $x[M_1] = \varepsilon$. Then $x = (x[M_1], x[M_2]) \in V(A, \lambda(A_{jj}))$.

Suppose now $x \in V(A, \lambda)$.

If $\lambda = \varepsilon$ then A has an ε column, say k^{th}, thus $a_{kk} = \varepsilon$. Hence the 1×1 submatrix (a_{kk}) is a diagonal block in an FNF of A. In the corresponding decomposition of N one of the sets, say N_j, is $\{k\}$. The set $\{i; N_i \to N_j\} = \{j\}$ and the theorem statement follows.

If $\lambda > \varepsilon$ and $x \in V^+(A)$ then $\lambda = \lambda(A)$ (cf. Theorem 1.3) and the statement now follows from (5).

If $\lambda > \varepsilon$ and $x \notin V^+(A)$ then similarly as in the proof of Lemma 2.2 permute the rows and columns of A simultaneously so that $x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$, where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p, x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \leq p < n)$. Hence $A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$ and we can assume without loss of generality that both $A^{(11)}$ and $A^{(22)}$ are in an FNF and therefore also $\begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$ is in an FNF. Let

$$
\begin{aligned}
A^{(11)} &= \begin{pmatrix} A_{i_1 i_1} & \varepsilon & ... & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{i_s i_1} & A_{i_s i_2} & ... & A_{i_s i_s} \end{pmatrix} \\
A^{(22)} &= \begin{pmatrix} A_{i_{s+1} i_{s+1}} & \varepsilon & ... & \varepsilon \\ A_{i_{s+2} i_{s+1}} & A_{i_{s+2} i_{s+2}} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{i_q i_{s+1}} & A_{i_q i_{s+2}} & ... & A_{i_q i_q} \end{pmatrix}
\end{aligned}
$$

We have $\lambda = \lambda(A^{(22)}) = \lambda(A_{jj}) = \max_{i=s+1,...,q} \lambda(A_{ii})$ where $j \in \{s+1, ..., q\}$. It remains to say that if $N_i \to N_j$ then $i \in \{s+1, ..., q\}$. ■

Note that this Theorem has already been proved in [13] and [3]. Spectral properties of reducible matrices were also studied in [4]. Significant correlation exists between the max-algebraic spectral theory and that for non-negative matrices in linear algebra [21], [5], see also [20]. For instance the Frobenius normal form and accessibility between classes play a key role in both theories. The maximum cycle mean corresponds to the Perron root for irreducible (nonnegative) matrices and finite eigenvectors in max-algebra correspond to positive eigenvectors in the non-negative spectral theory. However there are also differences, see Remark 3.2 after Theorem 3.2 below.

Let A be in the FNF (3). If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$$

then A_{jj} (and also N_j or just j) will be called *spectral*. Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way round.

Corollary 2.1 *All initial classes of C_A are spectral.*

Proof. Initial classes have no predecessors and so the condition of the Theorem is satisfied. ■

Corollary 2.2 $\lambda(A) \in \Lambda(A)$ *for every matrix A.*

Proof. If A is in an FNF, say (3) , then $\lambda(A) = \max_{i=1,...,r} \lambda(A_{ii}) = \lambda(A_{jj})$ for some j and so the condition of the Theorem is satisfied. ■

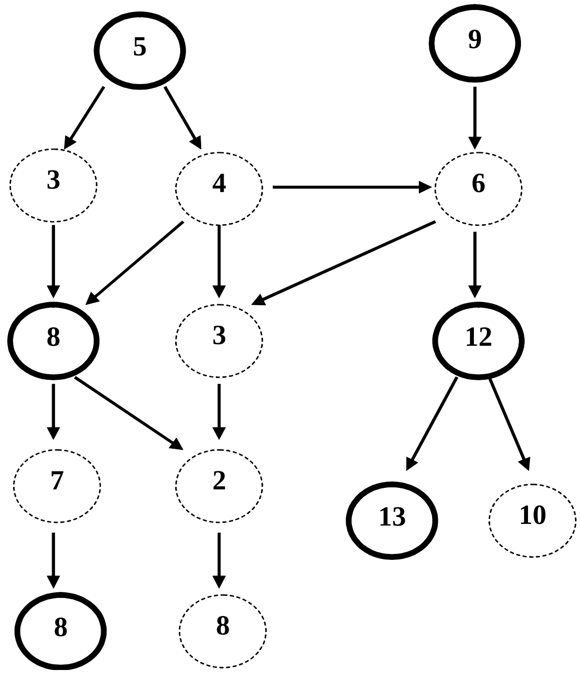

Figure 2: A condensation digraph with six spectral nodes

Corollary 2.3 $1 \leq |\Lambda(A)| \leq n$ *for every* $A \in \overline{\mathbb{R}}^{n\times n}$.

Proof. Follows from the previous corollary and from the fact that the number of classes of A is at most n. ■

Corollary 2.4 $V(A) = V(A, \lambda(A))$ *if and only if all initial classes have the same eigenvalue* $\lambda(A)$.

Proof. The eigenvalues of all initial classes are in $\Lambda(A)$ since all initial classes are spectral, hence all must be equal to $\lambda(A)$ if $\Lambda(A) = \{\lambda(A)\}$. On the other hand, if all initial classes have the same eigenvalue $\lambda(A)$, and λ is the eigenvalue of any spectral class then

$$\lambda \geq \lambda(A) = \max_i \lambda(A_{ii})$$

since there is a path from some initial class to this class and thus $\lambda = \lambda(A)$. ■

Figure 2 shows a condensation digraph with 14 classes including two initial classes and four final ones. The numbers indicate the eigenvalues of the corresponding classes. Six bold classes are spectral, the others are not.

3 Finding All Eigenvectors

Note that the unique eigenvalues of all classes (that is diagonal blocks of an FNF) can be found in $O(n^3)$ time by applying Karp's algorithm (see Section 1) to each block. The condition for identifying all spectral submatrices in an FNF provided in Theorem 2.2 enables us to find them in $O(r^2) \leq O(n^2)$ time by applying standard reachability algorithms to C_A.

Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in the FNF (3), $N_1, ..., N_r$ be the classes of A and $R = \{1, ..., r\}$. Suppose $\lambda \in \Lambda(A), \lambda > \varepsilon$ and denote $I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i$ spectral$\}$. Similarly as in Section 1 we denote $\Gamma(\lambda^{-1} \otimes A) = (g_{ij}) = (g_1, ..., g_n)$. Note that $\Gamma(\lambda^{-1} \otimes A)$ may now include entries equal to $+\infty$. Let us denote

$$E(\lambda) = \bigcup_{i\in I(\lambda)} E(A_{ii}) = \{j \in N; g_{jj} = 0, j \in \bigcup_{i\in I(\lambda)} N_i\}.$$

Two nodes i and j in $E(\lambda)$ are called λ - *equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle of cycle mean λ.

Theorem 3.1 *Suppose $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \Lambda(A), \lambda > \varepsilon$. Then $g_j \in \overline{\mathbb{R}}^n$ for all $j \in E(\lambda)$ and a basis of $V(A, \lambda)$ can be obtained by taking one g_j for each $\sim_\lambda$ equivalence class.*

Proof. Let us denote $M = \bigcup_{i\in I(\lambda)} N_i$. By Lemma 2.1 we may assume without loss of generality that A is of the form

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & A[M, M] \end{pmatrix}$$

Hence

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & C \end{pmatrix}$$

where $C = \Gamma((\lambda(A[M, M]))^{-1} \otimes A[M, M])$, and the theorem statement now follows by Theorems 1.1 and 1.2 since $\lambda = \lambda(A[M, M])$ and thus $\sim_\lambda$ equivalence for A is identical with $\sim$ equivalence for $A[M, M]$. ■

Corollary 3.1 *A basis of $V(A, \lambda)$ for $\lambda \in \Lambda(A)$ can be found using $O(n^3)$ operations and we have*

$$V(A, \lambda) = \{\Gamma(\lambda^{-1} \otimes A) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \text{ for all } j \notin E(\lambda)\}.$$

Alternatively, it follows from the proofs of Lemma 2.2 and Theorem 2.2 that $V(A, \lambda)$ can also be found as follows: If $I(\lambda) = \{j\}$ then define

$$M_2 = \bigcup_{N_i \to N_j} N_i, M_1 = N - M_2.$$

Hence

$$V(A,\lambda) = \{x; x[M_1] = \varepsilon, x[M_2] \in V^+(A[M_2, M_2])\}.$$

If the set $I(\lambda)$ consists of more than one index then the same process has to be repeated for each nonempty subset of $I(\lambda)$ that is for each $J \subseteq I(\lambda), J \neq \emptyset$ we set $S = \bigcup_{j \in J} N_j$ and

$$M_2 = \bigcup_{N_i \to S} N_i, M_1 = N - M_2.$$

Theorem 3.2 *$V^+(A) \neq \emptyset$ if and only if $\lambda(A)$ is the eigenvalue of all final classes (in all superblocks).*

Proof. The set M_1 in the above construction must be empty to obtain a finite eigenvector, hence a class in S must be reachable from every class of its superblock. This is only possible if S is the set of all final classes since no class is reachable from a final class (other than the final class itself). Conversely, if all final classes have the same eigenvalue $\lambda(A)$ then for $\lambda = \lambda(A)$ the set S contains all the final classes, they are reachable from all classes of their superblocks, and consequently $M_1 = \emptyset$, yielding a finite eigenvector. ■

Corollary 3.2 *$V^+(A) = \emptyset$ if and only if a final class has eigenvalue less than $\lambda(A)$.*

Remark 3.1 *Note that a final class with eigenvalue less than $\lambda(A)$ may not be spectral and so $\Lambda(A) = \{\lambda(A)\}$ is possible even if $V^+(A) = \emptyset$. For instance in the case of*

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\lambda(A) = 1,$ but $V^+(A) = \emptyset$.

Remark 3.2 *Following the terminology of non-negative matrices we say that a class is basic if its eigenvalue is $\lambda(A)$. It folows from Theorem 3.2 that $V^+(A)$ $\neq \emptyset$ if basic classes and final classes coincide. Obviously this requirement is not necessary, which is in contrast to the spectral theory of nonnegative matrices where for A to have a positive eigenvector it is necessary and sufficient that basic classes (that is those whose eigenvalue is the Perron root) are exactly the final classes.*

Remark 3.3 *By Corollary 2.2 for any matrix A at least one of the sets $V(A, \lambda(A)) - V^+(A, \lambda(A))$, $V^+(A, \lambda(A))$ is nonempty. We had*

$$V(A, \lambda(A)) - V^+(A, \lambda(A)) \neq \emptyset = V^+(A, \lambda(A))$$

in the previous example. For any irreducible matrix we have

$$V(A, \lambda(A)) - V^+(A, \lambda(A)) = \emptyset \neq V^+(A, \lambda(A))$$

and both sets are nonempty if for instance $A = I$.

4 Robustness of matrices

Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ and $x \in \overline{\mathbb{R}}^n$. The *orbit* of A with *starting vector* x is the sequence $O(A, x) = \{A^r \otimes x; r = 0, 1, ...\}$. Let $V'(A) = V(A) - \{\varepsilon\}$.

Let

$$T(A) = \{x \in \overline{\mathbb{R}}^n; O(A, x) \cap V'(A) \neq \emptyset\}.$$

Obviously,

$$V'(A) \subseteq T(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$$

holds for every matrix $A \in \overline{\mathbb{R}}^{n \times n}$.

It may happen that $T(A) = V'(A)$, for instance when A is the irreducible matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: Here $\lambda(A) = 0$ and by Theorem 1.3

$$V'(A) = \{\alpha \otimes (0, 0)^T; \alpha \in \mathbb{R}\}.$$

Since

$$A \otimes \begin{pmatrix} a \\ b \end{pmatrix} = (\max(a - 1, b), \max(a, b - 1))^T,$$

we have that $A \otimes \begin{pmatrix} a \\ b \end{pmatrix} \in V'(A)$ if and only if $a = b$, that is $A \otimes x \in V'(A)$ if and only if $x \in V'(A)$. Hence $T(A) = V'(A)$.

$T(A)$ may also be different from both $V'(A)$ and $\overline{\mathbb{R}}^n - \{\varepsilon\}$: Consider the irreducible matrix

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Here $\lambda(A) = 0$ and $x = (-2, -2, 0)^T \notin V'(A)$, but $A \otimes x = (-1, -1, 0)^T \in V'(A)$, showing that $T(A) \neq V'(A)$. At the same time if $y = (0, -1, 0)^T$ then $A^k \otimes y$ is y for k even and $(-1, 0, 0)^T$ for k odd, showing that $y \notin T(A)$.

Definition 4.1 *If $T(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$ then A is called robust.*

Hence A is robust if and only if for every $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ we have $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some positive integer k and $\lambda \in \Lambda(A)$. It is easily proved that if $A \sim B$ then A is robust if and only if B is robust. Therefore we may without loss of generality investigate robustness of matrices arising from a given matrix by a simultaneous permutation of the rows and columns.

Now we present some characterizations of robust matrices. First we observe that matrices with an ε column are not robust.

Following the terminology introduced in [11] we say that A is *column R-astic* if it has no ε column.

Lemma 4.1 *If $A \in \overline{\mathbb{R}}^{n\times n}$ is column $\mathbb{R}$-astic and $x \neq \varepsilon$ then $A^k \otimes x \neq \varepsilon$ for every k. Hence if $A \in \overline{\mathbb{R}}^{n\times n}$ is column $\mathbb{R}$-astic then A^k is column $\mathbb{R}$-astic for every k. This is true in particular when A is irreducible and $n > 1$.*

Proof. Immediate from definition. ■

Note that every node of a non-trivial strongly connected digraph has at least one incoming arc and so every irreducible $n\times n$ matrix ($n > 1$) is column $\mathbb{R}$-astic (but not conversely).

We say that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is *ultimately periodic of period* p if there is a natural number p such that the following holds for some $\lambda \in \mathbb{R}$ and k_0 natural:

$$A^{k+p} = \lambda^p \otimes A^k \text{ for all } k \geq k_0.$$

If p is the smallest natural number with this property then we call p the *period* of A and denote it as $p(A)$. If A is not ultimately periodic then we set $p(A) = +\infty$. It is easily seen that $\lambda = \lambda(A)$ and every column of A^k is in $V(A^p, \lambda^p)$ if $p = p(A) < +\infty$ and A is column $\mathbb{R}$-astic. Robustness of irreducible matrices was studied in [7] and we now mention some results of that paper before we proceed with the reducible case. Note that if A is the 1×1 matrix (ε) then A is irreducible, $p(A) = 1$ but A is not robust. This is an exceptional case that has to be excluded in the statements that follow.

Theorem 4.1 *[7] Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible, $A \neq \varepsilon$. Then A is robust if and only if $p(A) = 1$.*

Corollary 4.1 *[7] Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible, $A \neq \varepsilon$. If $p(A) = 1, x \neq \varepsilon$ then $A^k \otimes x$ is finite for all sufficiently big k.*

It was shown in [7] how the next statement follows from the results in [6], Theorem 3.4.5.

Theorem 4.2 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be irreducible. Then A^k is irreducible for every $k = 1, 2, ...$ if and only if the lengths of all cycles in D_A are co-prime.*

Theorem 4.3 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}, A \neq \varepsilon$ be irreducible. Then A is robust if and only if the eigenspaces of A^k coincide for every $k = 1, 2, ...$*

Previous results are closely related to the famous "Cyclicity Theorem", Theorem 4.4 below. For this we need to introduce a few more concepts: Let D' be a maximal strongly connected subdigraph of a digraph D. Then D' is called a *strongly connected component* of D and the greatest common divisor of all directed cycles in D' is called the *cyclicity* of D', notation $\sigma(D')$. By definition $\sigma(D') = 1$ if D' consists only of a single node. The cyclicity of D is the least common multiple of cyclicities of all strongly connected components of D.

Theorem 4.4 *Every irreducible matrix A is ultimately periodic and $p(A) = \sigma(C(A))$.*

Corollary 4.2 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be irreducible and robust. Then A^k is irreducible for every $k = 1, 2, ...$*

Proof. If the lengths of all critical cycles in D_A are co-prime then also the lengths of all cycles are co-prime. The rest follows from Theorem 4.2. ■

Note that the "if" statement of Theorem 4.1 follows immediately from Theorem 4.4.

First part of Theorem 4.4 was proved for finite matrices in [11]. A proof of the whole statement was presented in [8], see also [9] for an overview without proofs. A proof in a more general setting covering the case of finite matrices is given in [19]. The irreducible case is also proved in [1], [17], [2] and [14]. Note that a different generalization to the reducible case is studied in [15].

We now continue by studying robustness of reducible matrices. Theorem 4.1

can straightforwardly be generalized to a class of reducible matrices:

Theorem 4.5 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be column $\mathbb{R}$-astic and $|\Lambda(A)| = 1$ (that is $\Lambda(A) = \{\lambda(A)\}$). Then A is robust if and only if $p(A) = 1$.*

Proof. Let $p(A) = 1, x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ and $k \geq k_0$. Then $A^k \otimes x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ by Lemma 4.1, $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ and so $A^k \otimes x \in V(A, \lambda)$ and $\lambda = \lambda(A)$. Hence A is robust and all columns of A^k are eigenvectors of A.

Now let A be robust and all columns of A^{k_0} be eigenvectors of A corresponding to the unique eigenvalue $\lambda(A)$. Then $A \otimes A^{k_0} = \lambda(A) \otimes A^{k_0}$ and thus $A \otimes A^k = \lambda(A) \otimes A^k$ for all $k \geq k_0$. So $p(A) = 1$. ■

We will now characterise robust reducible matrices in general - we start with two lemmas.

Lemma 4.2 *If $A \in \overline{\mathbb{R}}^{n\times n}$ is robust then $\varepsilon \notin \Lambda(A)$.*

Proof. If $\varepsilon \in \Lambda(A)$ then by Lemma 4.1 some column, say kth is ε. Take $x \in \overline{\mathbb{R}}^n$ so that $x_k = 0$ and $x_j = \varepsilon$ for $j \neq k$. Then $A^k \otimes x = \varepsilon$ for every k and thus $A^k \otimes x$ is never an eigenvector. ■

Recall that if $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is in the FNF (3) and $N_1, ..., N_r$ are the classes of A then we have denoted $R = \{1, ..., r\}$. If $i \in R$ then we now also denote $T_i = \{k \in R; N_k \longrightarrow N_i\}$ and $M_i = \bigcup_{j\in T_i} N_j$. A class N_i of A is called *trivial* if N_i contains only one index, say k, and $a_{kk} = \varepsilon$.

Lemma 4.3 *If every non-trivial class of $A \in \overline{\mathbb{R}}^{n\times n}$ has eigenvalue 0 and period 1 then $A^{k+1} = A^k$ for some k.*

Proof. We prove the statement by induction on the number of classes.

If A has only one class then either this class is trivial or A is irreducible. In both cases the statement follows immediately.

If A has at least two classes then by Lemma 2.1 we can assume without loss of generality:

$$A = \begin{pmatrix} A_{11} & \varepsilon \\ A_{21} & A_{22} \end{pmatrix}$$

and thus

$$A^k = \begin{pmatrix} A_{11}^k & \varepsilon \\ B_k & A_{22}^k \end{pmatrix}$$

where

$$B_k = \sum_{i+j=k-1}^{\oplus} A_{22}^i \otimes A_{21} \otimes A_{11}^j.$$

By the induction hypothesis there are k_1 and k_2 such that

$$A_{11}^{k_1+1} = A_{11}^{k_1} \text{ and } A_{22}^{k_2+1} = A_{22}^{k_2}.$$

It is sufficient now to prove that

$$B_k = \sum^{\oplus} \left\{ A_{22}^i \otimes A_{21} \otimes A_{11}^j; i \leq k_2, j \leq k_1, i = k_2 \text{ or } j = k_1 \right\} \tag{6}$$

holds for all $k \geq k_1 + k_2 + 1$.

For all i, j we have

$$A_{22}^i \otimes A_{21} \otimes A_{11}^j = A_{22}^{i'} \otimes A_{21} \otimes A_{11}^{j'}$$

where $i' = \min(i, k_2), j' = \min(j, k_1)$. If $i + j + 1 = k \geq k_1 + k_2 + 1$ then either $i \geq k_2$ or $j \geq k_1$. Hence either $i' = k_2$ or $j' = k_1$ and therefore $\leq$ in (4) follows. For $\geq$ let $i = k_2$ (say) and $j \leq k_1$. Since $k \geq k_1 + k_2 + 1 \geq j + i + 1$, we have $k - j - 1 \geq i = k_2$ and thus

$$A_{22}^i \otimes A_{21} \otimes A_{11}^j = A_{22}^{k-j-1} \otimes A_{21} \otimes A_{11}^j \leq B_k.$$

∎

Theorem 4.6 *Let $A \in \overline{\mathbb{R}}^{n \times n}, A \neq \varepsilon$ be in the FNF (3), $N_1, ..., N_r$ be the classes of A and $R = \{1, ..., r\}$. Then A is robust if and only if the following hold:*

1. *All non-trivial classes $N_1, ..., N_r$ are spectral.*
2. *If $i, j \in R, N_i, N_j$ are non-trivial and $i \notin T_j$ and $j \notin T_i$ then $\lambda(N_i) = \lambda(N_j)$.*
3. *$p(A_{jj}) = 1$ for all $j \in R$.*

Proof. If $r = 1$ then A is irreducible and the statement follows by Theorem 4.1. We will therefore assume $r \geq 2$ in this proof.

Let A be robust.

1. Let $i \in R, A_{ii} \neq \varepsilon$ and $x \in \overline{\mathbb{R}}^n$ be defined by taking any $x_s \in \mathbb{R}$ for $s \in M_i$ and $x_s = \varepsilon$ for $s \notin M_i$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$. Let $z = A^k \otimes x$. Then $z[M_i]$ is finite since $A[M_i]$ has no ε row and $A[M_i] \otimes z[M_i] = (A \otimes z)[M_i] = \lambda \otimes z[M_i]$ and thus $z[M_i] \in V^+(A[M_i])$. By Lemma 4.2 $\lambda > \varepsilon$ and so by Theorem 1.3 then $\lambda(N_t) \leq \lambda(N_i)$ for all $t \in T_i$. Hence N_i is spectral.

2. Suppose $i, j \in R, N_i, N_j$ are non-trivial and $i \notin T_j$, $j \notin T_i$. Let $x \in \overline{\mathbb{R}}^n$ be defined by taking any $x[N_i] \in V^+(A[N_i]), x[N_j] \in V^+(A[N_j])$ and $x_s = \varepsilon$ for $s \in N - N_i \cup N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$. Denote $z = A^k \otimes x$. Then $z[N_j]$ is finite. Since $i \notin T_j$ we have $a_{uv} = \varepsilon$ for all $u \in N_i$ and $v \in N_j$. Hence

$$\lambda \otimes z[N_j] = (A \otimes z)[N_j] = A[N_j] \otimes z[N_j]$$

and so by Theorem 1.3 $\lambda(N_j) = \lambda$. Similarly it is proved that $\lambda(N_i) = \lambda$.

3. Let $j \in R$ and $A[N_j] \neq \varepsilon$ (otherwise the statement follows trivially). Let $x \in \overline{\mathbb{R}}^n$ be any vector such that $x \neq \varepsilon$ and $x_s = \varepsilon$ for $s \notin N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$. Let $z = A^k \otimes x$. Since $z[N_j] = (A[N_j])^k \otimes x[N_j]$ we may assume without loss of generality that $z[N_j]$ is finite due to Corollary 4.1. At the same time $A[N_j] \otimes z[N_j] = (A \otimes z)[N_j] = \lambda \otimes z[N_j]$ and thus $z[N_j] \in V(A[N_j])$. Hence $A[N_j]$ is irreducible and robust. Thus by Theorem 4.1 $p\left(A[N_j]\right) = p(A_{jj}) = 1$.

Suppose now that conditions 1.-3. are satisfied. We prove then that A is robust by induction on the number of classes of A. As already observed at the beginning of this proof the case $r = 1$ follows from Theorem 4.1. Suppose now that $r \geq 2$ and let $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$. Let

$$U = \{i \in N; (\exists j)\, i \longrightarrow j, x_j \neq \varepsilon\}.$$

We have

$$\left(A^k \otimes x\right)[U] = \left(A[U,U]\right)^k \otimes x[U]$$

and

$$\left(A^k \otimes x\right)_i = \varepsilon$$

for $i \notin U$. Therefore we may assume without loss of generality that $U = N$. Let M be a final class in C_A, clearly $x[M] \neq \varepsilon$ by the definition of U. Let us denote

$$\begin{aligned} S &= \{i \in N; (\exists j \in M)\, (i \longrightarrow j)\} \\ S' &= N \setminus S. \end{aligned}$$

By Lemma 2.1 we may assume without loss of generality that

$$A = \begin{pmatrix} A_{11} & \varepsilon & \varepsilon \\ A_{21} & A_{22} & A_{23} \\ \varepsilon & \varepsilon & A_{33} \end{pmatrix}$$

where the individual blocks correspond (in this order) to the sets $M, S \setminus M$ and S' respectively. Let us define $x^k = A^k \otimes x$ for all integers $k \geq 0$. We also set

$$\begin{aligned} x_1^k &= x^k[M] \\ x_2^k &= x^k[S \setminus M] \\ x_3^k &= x^k[S'] \end{aligned}$$

Obviously,

$$\begin{aligned} x_1^{k+1} &= A_{11} \otimes x_1^k \\ x_2^{k+1} &= A_{21} \otimes x_1^k \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^k \\ x_3^{k+1} &= A_{33} \otimes x_3^k \end{aligned}$$

Assume first that M is non-trivial. Then $\lambda(A_{11}) \neq \varepsilon$ and by taking (if necessary) $(\lambda(A_{11}))^{-1} \otimes A$ instead of A, we may assume without loss of generality that $\lambda(A_{11}) = 0$. By assumption 3 and Theorem 4.4 we have $A_{11}^{k_1+1} = A_{11}^{k_1}$ for some k_1. By assumption 2 every class of A_{33} has eigenvalue 0. Since each of these classes has also period 1 by assumption 3, it follows from Lemma 4.3 that $A_{33}^{k_3+1} = A_{33}^{k_3}$ for some k_3. We may also assume without loss of generality that

$$\begin{aligned} x_1^0 &= x_1^1 = x_1^2 = ... \\ x_3^0 &= x_3^1 = x_3^2 = ... \end{aligned}$$

Therefore

$$x_2^{k+1} = A_{21} \otimes x_1^0 \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^0.$$

Let $v = A_{21} \otimes x_1^0 \oplus A_{23} \otimes x_3^0$. We deduce that

$$x_2^k = A_{22}^k \otimes x_2^0 \oplus \left(A_{22}^{k-1} \oplus ... \oplus A_{22}^0\right) \otimes v \tag{7}$$

for all k. Moreover, $\lambda(A_{22}) \leq \lambda(A_{11}) = 0$ since M is spectral by assumption 1. Hence

$$A_{22}^{k-1} \oplus ... \oplus A_{22}^0 = \Gamma\left(A_{22}\right)$$

for all $k \geq n$. Note that x_1^0 is finite as an eigenvector of the irreducible matrix A_{11}. Also, since every node in S has access to M, the vector $\Gamma\left(A_{22}\right) \otimes A_{21} \otimes x_1^0$ is finite and hence also $\Gamma\left(A_{22}\right) \otimes v$ is finite. If $\lambda(A_{22}) < 0$ then $A_{22}^k \otimes x_2^0 \longrightarrow -\infty$ as $k \longrightarrow \infty$ and we deduce that $x_2^k = \Gamma\left(A_{22}\right) \otimes v$ for all k big enough. If $\lambda(A_{22}) = 0$ then

$$A_{22}^{k_2+1} = A_{22}^{k_2}$$

by the induction hypothesis and thus

$$x_2^k = A_{22}^{k_2} \otimes x_2^0 \oplus \Gamma\left(A_{22}\right) \otimes v$$

for all $k \geq \max(k_1, k_2, k_3)$.

It remains to consider the case when A_{11} is trivial. Then $x_1^k = \varepsilon$ for all $k \geq 1$ and we have

$$\begin{pmatrix} x_2^{k+1} \\ x_3^{k+1} \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ \varepsilon & A_{33} \end{pmatrix} \otimes \begin{pmatrix} x_2^k \\ x_3^k \end{pmatrix}$$

for all $k \geq 1$. We apply the induction hypothesis to the matrix

$$\begin{pmatrix} A_{22} & A_{23} \\ \varepsilon & A_{33} \end{pmatrix}$$

and deduce that $x^{k+1} = x^k$ for k sufficiently big. This completes the proof. ■

Example 4.1 *Let* $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$, *thus* $r = 3, \Lambda(A) = \{0, 1, 2\}, N_j = \{j\}, j = 1, 2, 3$. *If* $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, *then* $O(A, x)$ *is*

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}, ...$$

which obviously will never reach an eigenvector. The reason is that $1 \notin T_2, 2 \notin T_1$ *but* $\lambda(N_1) \neq \lambda(N_2)$.

Example 4.2 *Let* $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$, *thus* $r = 3, \Lambda(A) = \{0, 2\}, N_j = \{j\}, j = 1, 2, 3$. *This matrix is robust since both non-trivial classes* (N_1 *and* N_3) *are spectral,* $p(A_{ii}) = 1$ $(i = 1, 2, 3)$ *and there are no non-trivial classes* N_i, N_j *such that* $i \notin T_j$ *and* $j \notin T_i$. *Indeed, if* $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, *then* $O(A, x)$ *is*

$$\begin{pmatrix} 2 \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ \varepsilon \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ \varepsilon \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ \varepsilon \\ 6 \end{pmatrix}, ...$$

hence an eigenvector is reached in the first step.

Note that requirements 1. to 3. of Theorem 4.6 imply that every robust matrix A either has only one superblock or $|\Lambda(A)| = 1$. Obviously this restricts the concept of robustness for reducible matrices quite significantly. Therefore we aim to introduce a modification of robustness and provide a criterion which will enable us to characterise a wider class of matrices displaying robustness properties reflecting the rich spectral structure of reducible matrices.

We start with a simple observation.

Lemma 4.4 *Let* $A = \begin{pmatrix} A' & \varepsilon \\ ... & A[M] \end{pmatrix}$ *be column* $\mathbb{R}$*-astic,* $x \in \overline{\mathbb{R}}^n$, $M \subseteq N$ *and* $y = A^k \otimes x$. *If* $x[N - M] = \varepsilon$ *then* $y[N - M] = \varepsilon$.

Proof. Straightforward. ■

Let $x \in \overline{\mathbb{R}}^n$. The set $\{j \in N; x_j > \varepsilon\}$ is called the *support* of x, notation $s(x)$. Lemma 4.4 is implying that if M is the support of an eigenvector and $s(x) \subseteq M$ then $s(A^k \otimes x) \subseteq M$ for all positive integers k. This motivates the following definitions:

Definition 4.2 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be in an FNF. Then $M \subseteq N$ is called regular if for some λ there is an $x \in V(A, \lambda)$ with $x[M]$ finite and $x[N-M] = \varepsilon$. We also denote $\lambda = \lambda(M)$.*

Remark 4.1 *Even if M is regular there still may exist an $x \in V(A, \lambda(M))$ with $x_j = \varepsilon$ for some $j \in M$.*

Since for a given matrix the finiteness structure of all eigenvectors is well described (see Section 3) we aim to characterise matrices for which an eigenvector in $V(A, \lambda(M))$ for a given regular set M is reached with any starting vector whose support is a subset of M.

It follows from the decription of $V(A)$ in Section 3 that M is regular if and only if there exist spectral indices $i_1, ..., i_s$ for some s such that $M = \{i \in N; i \to N_{i_1} \cup ... \cup N_{i_s}\}$.

Let $M \subseteq N$. We denote

$$\overline{\mathbb{R}}^n(M) = \{x \in \overline{\mathbb{R}}^n - \{\varepsilon\}; (\forall j \in N - M)(x_j = \varepsilon)\ \}.$$

Definition 4.3 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be a column $\mathbb{R}$-astic matrix in an FNF and $M \subseteq N$ be regular. Then A will be called M-robust if*

$$(\forall x \in \overline{\mathbb{R}}^n(M))(\exists k) A^k \otimes x \in V(A, \lambda(M)).$$

Theorem 4.7 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be a column $\mathbb{R}$-astic matrix in an FNF, $M \subseteq N$ be regular and $B = A[M]$. Then A is $M-$robust if and only if $p(B) = 1$.*

Proof. Without loss of generality let $A = \begin{pmatrix} A[N-M] & \varepsilon \\ \cdots & B \end{pmatrix}$.

Supose that A is M-robust. Take $x = A_j, j \in M$. Then $x \in \overline{\mathbb{R}}^n(M)$ because A (and therefore also B) is column $\mathbb{R}$-astic and there is a k_j such that $A^k \otimes A_j \in V(A, \lambda(M))$ for all $k \geq k_j$. Since $A_j = \begin{pmatrix} \varepsilon \\ A_j[M] \end{pmatrix}$, we have

$$A \otimes \left(A^k \otimes A_j\right) = \begin{pmatrix} \varepsilon \\ B \otimes \left(B^k \otimes A_j[M]\right) \end{pmatrix} = \lambda(M) \otimes \begin{pmatrix} \varepsilon \\ B^k \otimes A_j[M] \end{pmatrix}.$$

Hence, for $k \geq \max_{j \in M} k_j$ there is

$$B^{k+2} = \lambda(M) \otimes B^{k+1}$$

that is $p(B) = 1$ with $\lambda = \lambda(M)$.

Suppose now $B^{k+1} = \lambda \otimes B^k$ for some λ and for all $k \geq k_0$. If the FNF of B is

$$B = \begin{pmatrix} B_1 & & \varepsilon \\ \vdots & \ddots & \\ \cdot & \cdots & B_r \end{pmatrix}$$

then

$$B^k = \begin{pmatrix} B_1^k & & \varepsilon \\ \vdots & \ddots & \\ \cdot & \cdots & B_r^k \end{pmatrix}$$

and so $B_i^{k+1} = \lambda \otimes B_i^k$ $(i = 1, ..., r)$. But since every B_i is irreducible, $\lambda = \lambda(B_i) = \lambda(M)$ $(i = 1, ..., r)$. Let $M = M_1 \cup ... \cup M_r$ be the partition of M determined by the FNF of B. Let $x \in \overline{\mathbb{R}}^n(M), x = (x[N-M] = \varepsilon, x[M_1], ..., x[M_r])$ and let

$$s = \min\{i; x[M_i] \neq \varepsilon\}.$$

Denote $y = A^k \otimes x, y = (y[N-M], y[M_1], ..., y[M_r])$. Clearly, $y[N-M] = \varepsilon$ and

$$y[M_s] = B^k \otimes x[M_s] \neq \varepsilon$$

since B_s is irreducible (note that using Corollary 4.1 it would be possible to prove here that $y[M_i]$ is finite for all $i \geq s$). Hence $y \in \overline{\mathbb{R}}^n(M)$. At the same time

$$B^{k+1} \otimes x[M] = \lambda \otimes B^k \otimes x[M]$$

and

$$y = \begin{pmatrix} \varepsilon \\ B^k \otimes x[M] \end{pmatrix}.$$

Therefore

$$A \otimes y = \begin{pmatrix} \varepsilon \\ B \otimes B^k \otimes x[M] \end{pmatrix} = \lambda(M) \otimes \begin{pmatrix} \varepsilon \\ B^k \otimes x[M] \end{pmatrix} = \lambda(M) \otimes y.$$

We conclude that $y \in V(A, \lambda(M))$. ■

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